

A Class of LULU Operators on Multi-Dimensional Arrays

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Abstract

The LULU operators for sequences are extended to multi-dimensional arrays via the morphological concept of connection in a way which preserves their essential properties, e.g. they are separators and form a four element fully ordered semi-group. The power of the operators is demonstrated by deriving a total variation preserving discrete pulse decomposition of images.

Keywords: LULU, connection, separator, discrete pulse transform, total variation.

1 Introduction

The LULU operators and the associated Discrete Pulse Transform developed during the last two decades or so are an important contribution to the theory of the nonlinear multi-resolution analysis of sequences. The basics of the theory as well as the most significant results until 2005 are published in the monograph [13]. For more recent developments and applications see [1], [4], [7], [8], [14]. Central to the theory is the concept of *separator*. This concept is defined in [13] only for operators on sequences due to the context of the book. However, it is meaningful in more general settings. In fact, some of the axioms have been used earlier, e.g. see [18], for functions on arbitrary domains. We will give the definition of separator for operators on real functions defined on a domain with a group structure.

Let a Ω be an abelian group. Denote by $\mathcal{A}(\Omega)$ the vector lattice of all real functions defined on Ω with respect to the usual point-wise defined addition, scalar multiplication and partial order. For every $a \in \Omega$ the operator $E_a : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ given by

$$E_a(f)(x) = f(x + a), \quad x \in \Omega,$$

is called a shift operator.

Definition 1 An operator $P : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega)$ is called a **separator** if

- (i) $P \circ E_a = E_a \circ P, a \in \Omega;$
- (ii) $P(f + c) = P(f) + c, f, c \in \mathcal{A}(\Omega), c - \text{constant function};$
- (iii) $P(\alpha f) = \alpha P(f), \alpha \in \mathbb{R}, \alpha \geq 0, f \in \mathcal{A}(\Omega);$
- (iv) $P \circ P = P;$ (Idempotence)
- (v) $(id - P) \circ (id - P) = id - P.$ (Co-idempotence)

Here id denotes the identity operator and the operator $id - P$ is defined in terms of the point-wise linear operations for the operators on $\mathcal{A}(\Omega)$, that is, $(id - P)(f) = f - P(f)$. The first two axioms in Definition 1 and partially the third one were first introduced as required properties of nonlinear smoothers by Mallows, [9]. Rohwer further made the concept of a smoother more precise by using the properties (i)–(iii) as a definition of this concept. The axiom (iv) is an essential requirement for what is called a *morphological filter*, [18], [19], [21]. In fact, a morphological filter is exactly a syntone operator which satisfies (iv). Let us recall that an operator P is called *syntone* if

$$f \leq g \implies P(f) \leq P(g), \quad f, g \in \mathcal{A}(\Omega).$$

The co-idempotence axiom (v) in Definition 1 was introduced by Rohwer in [13], where it is also shown that it is an essential requirement for operators extracting signal from a sequence.

The LULU theory was developed for sequences, that is, the case $\Omega = \mathbb{Z}$. Given a bi-infinite sequence $\xi = (\xi_i)_{i \in \mathbb{Z}}$ and $n \in \mathbb{N}$ the basic LULU operators L_n and U_n are defined as follows

$$(L_n \xi)_i = \max\{\min\{\xi_{i-n}, \dots, \xi_i\}, \dots, \min\{\xi_i, \dots, \xi_{i+n}\}\}, \quad i \in \mathbb{Z}. \quad (1)$$

$$(U_n \xi)_i = \min\{\max\{\xi_{i-n}, \dots, \xi_i\}, \dots, \max\{\xi_i, \dots, \xi_{i+n}\}\}, \quad i \in \mathbb{Z}. \quad (2)$$

It is shown in [13] that for every $n \in \mathbb{N}$ the operators L_n and U_n as well as their compositions are syntone separators. Hence they are an appropriate tool for signal extraction. Furthermore, these operators form the so called strong LULU semi-group. This a four element semi-group with respect to composition, see Table 1, which is fully ordered with respect to the usual point-wise defined order

$$P \leq Q \iff P(f) \leq Q(f), \quad f \in \mathcal{A}(\Omega). \quad (3)$$

We have

$$L_n \leq U_n \circ L_n \leq L_n \circ U_n \leq U_n. \quad (4)$$

Let us recall that, according to the well known theorem of Matheron [10], in general, two ordered morphological operators generate a six element semi-group which is only partially ordered.

The power of the LULU operators as separators is further demonstrated by their Total Variation Preservation property. Let $BV(\mathbb{Z})$ be the set of sequences with bounded variation, that is,

$$BV(\mathbb{Z}) = \{\xi \in \mathcal{A}(\mathbb{Z}) : \sum_{i \in \mathbb{Z}} |\xi_i - \xi_{i+1}| < \infty\}.$$

Total Variation of a sequence $\xi \in BV(\mathbb{Z})$ is given by $TV(\xi) = \sum_{i \in \mathbb{Z}} |\xi_i - \xi_{i+1}|$.

	L_n	U_n	$U_n \circ L_n$	$L_n \circ U_n$
L_n	L_n	$L_n \circ U_n$	$U_n \circ L_n$	$L_n \circ U_n$
U_n	$U_n \circ L_n$	U_n	$U_n \circ L_n$	$L_n \circ U_n$
$U_n \circ L_n$	$U_n \circ L_n$	$L_n \circ U_n$	$U_n \circ L_n$	$L_n \circ U_n$
$L_n \circ U_n$	$U_n \circ L_n$	$L_n \circ U_n$	$U_n \circ L_n$	$L_n \circ U_n$

Table 1: LULU semi-group

Definition 2 *An operator $P : BV(\mathbb{Z}) \rightarrow BV(\mathbb{Z})$ is called **total variation preserving** if*

$$TV(\xi) = TV(P(\xi)) + TV((id - P)(\xi)) , \quad \xi \in BV(\mathbb{Z}). \quad (5)$$

We should note that since TV is a semi-norm on $BV(\mathbb{Z})$ we always have

$$TV(\xi) \leq TV(P(\xi)) + TV((id - P)(\xi)).$$

Hence, the significance of the equality (5) is that the decomposition $f = A(f) + (id - A)(f)$ does not create additional total variation. In particular, this property is very important for the application of the LULU operators to discrete pulse decompositions of sequences.

The aim of this paper is to generalize the LULU operators to functions on \mathbb{Z}^d in such a way that their essential properties are preserved. In Section 2 the definitions of the basic operators L_n and U_n on $\mathcal{A}(\mathbb{Z}^d)$ are derived by using a strengthened form of the morphological concept of connection. Then we show that indeed these operators replicate the properties of the LULU operators for sequence. More precisely, we prove that: (i) they are separators (Section 2); (ii) their smoothing effect can be described in a similar way to the n -monotonicity of sequences (Section 3); (iii) they generate a four element fully ordered semi-group (Section 4). The developed theory can be applied to many problems of Image Analysis and it is the intention of the authors to research such applications in the future. However, as an illustration and demonstration of the power of this approach we apply the newly defined operators to deriving a total variation preserving discrete pulse decomposition of images. Noise removal and partial reconstructions are discussed in Section 6.

2 The basic operators L_n and U_n .

The definition of the operators L_n and U_n for sequences involves maxima and minima over sets of consecutive terms, thus, making an essential use of the fact that \mathbb{Z} is totally ordered. Since \mathbb{Z}^d , $d > 1$, is only partially ordered the concept of ‘consecutive’ does not make sense in this setting. Instead, we use the morphological concept of set connection, [19].

Definition 3 *Let B be an arbitrary non-empty set. A family \mathcal{C} of subsets of B is called a **connected class** or a **connection** on B if*

(i) $\emptyset \in \mathcal{C}$

- (ii) $\{x\} \in \mathcal{C}$ for all $x \in B$
- (iii) for any family $\{C_i : i \in \mathcal{C}\} \subseteq \mathcal{C}$

$$\bigcap_{i \in I} C_i \neq \emptyset \implies \bigcup_{i \in I} C_i \in \mathcal{C}$$

This definition generalizes the topological concept of connectivity to arbitrary sets including discrete sets like \mathbb{Z}^d . If a set C belongs to a connection \mathcal{C} then C is called *connected*.

It is clear from Definition 3 that a connection on \mathbb{Z}^d does not necessarily contain sets of every size. For example, $\{\emptyset\} \cup \{\{x\} : x \in \mathbb{Z}^d\}$ and $\{\emptyset\} \cup \{\{x\} : x \in \mathbb{Z}^d\} \cup \{\mathbb{Z}^d\}$ are connections on \mathbb{Z}^d but neither of them contain sets of finite size other than 0 and 1. In the definition of the operators L_n and U_n we need sets of every size. We assume that the set \mathbb{Z}^d is equipped with a connection \mathcal{C} which satisfies the following three conditions

- $\mathbb{Z}^d \in \mathcal{C}$ (6)

- For any $a \in \mathbb{Z}^d$, $E_a(C) \in \mathcal{C}$ whenever $C \in \mathcal{C}$ (7)

- If $V \subsetneq W$, $V, W \in \mathcal{C}$, then there exists $x \in W \setminus V$ such that $V \cup \{x\} \in \mathcal{C}$ (8)

The aim of the conditions (6)–(8) is to define a connection which is sufficiently rich in connected sets. This is demonstrated by the following property, which is obtained via iterative application of the property (8):

- Let $V \subsetneq W$, $V, W \in \mathcal{C}$. For every $k \in \mathbb{N}$ such that $\text{card}(V) < k < \text{card}(W)$ there exists $S \in \mathcal{C}$ such that $V \subseteq S \subseteq W$ and $\text{card}(S) = k$. (9)

As usual, $\text{card}(V)$ is the number of the elements in the set V , that is, the size of V . For $V \subseteq \mathbb{Z}^d$ we have $\text{card}(V) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. Given a point $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ we denote by $\mathcal{N}_n(x)$ the set of all connected sets of size $n + 1$, which contain point x , that is,

$$\mathcal{N}_n(x) = \{V \in \mathcal{C} : x \in V, \text{card}(V) = n + 1\}. \quad (10)$$

Now the operators L_n and U_n are defined on $\mathcal{A}(\mathbb{Z}^d)$ as follows.

Definition 4 Let $f \in \mathcal{A}(\mathbb{Z}^d)$ and $n \in \mathbb{N}$. Then

$$L_n(f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y), \quad x \in \mathbb{Z}^d, \quad (11)$$

$$U_n(f)(x) = \min_{V \in \mathcal{N}_n(x)} \max_{y \in V} f(y), \quad x \in \mathbb{Z}^d. \quad (12)$$

Let us first see that Definition 4 generalizes the definition of L_n and U_n for sequences. Suppose $d = 1$ and let \mathcal{C} be the connection on \mathbb{Z} generated by the pairs of consecutive numbers. Then all connected sets on \mathbb{Z} are sequences of consecutive integers and for any $i \in \mathbb{Z}$ we have

$$\mathcal{N}_n(i) = \{\{i-n, i-n+1, \dots, i\}, \{i-n+1, i-n+2, \dots, i+1\}, \dots, \{i, i+1, \dots, i+n\}\}$$

Hence for an arbitrary sequence ξ considered as a function on \mathbb{Z} the formulas (11) and (12) are reduced to (1) and (2), respectively.

Theorem 5 (Order Properties)

- a) $L_n \leq id \leq U_n$
- b) $f \leq g \implies (L_n(f) \leq L_n(g), U_n(f) \leq U_n(g))$
- c) $n_1 < n_2 \implies (L_{n_1} \geq L_{n_2}, U_{n_1} \leq U_{n_2})$

Proof. We will only prove the inequalities involving L_n since those involving U_n are proved similarly.

a) Let $f \in \mathcal{A}(\mathbb{Z}^d)$. For every $x \in \mathbb{Z}^d$ and $V \in \mathcal{N}_n(x)$ we have

$$\min_{y \in V} f(y) \leq f(x).$$

Hence

$$L_n(f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y) \leq f(x), \quad x \in \mathbb{Z}^d.$$

Therefore, $L_n(f) \leq f$, $f \in \mathcal{A}(\mathbb{Z}^d)$, which implies $L_n \leq id$.

b) Let $f \leq g$. For any $x \in \mathbb{Z}^d$ and $V \in \mathcal{N}_n(x)$, we have $\min_{y \in V} f(y) \leq \min_{y \in V} g(y)$.

Therefore

$$L_n(f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y) \leq \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} g(y) = L_n(g)(x), \quad x \in \mathbb{Z}^d.$$

c) Let $f \in \mathcal{A}(\mathbb{Z}^d)$. It follows from (10) that for every $x \in \mathbb{Z}^d$ and $V \in \mathcal{N}_{n_2}(x)$ there exists a set $W \in \mathcal{N}_{n_1}(x)$ such that $W \subseteq V$. Therefore

$$\min_{y \in V} f(y) \leq \min_{y \in W} f(y) \leq \max_{S \in \mathcal{N}_{n_1}(x)} \min_{y \in S} f(y) = L_{n_1}(f)(x).$$

Hence

$$L_{n_2}(f)(x) = \max_{V \in \mathcal{N}_{n_2}(x)} \min_{y \in V} f(y) \leq L_{n_1}(f)(x), \quad x \in \mathbb{Z}^d.$$

■

Theorem 6 For any $n \in \mathbb{N}$ the operators L_n and U_n are separators.

Proof. We will only verify the conditions (i)–(v) in Definition 1 for L_n since U_n is dealt with in a similar manner.

(i) Let $a \in \mathbb{Z}^d$ and $f \in \mathcal{A}(\mathbb{Z}^d)$. Using the property (7), for every $x \in \mathbb{Z}^d$ we have

$$\mathcal{N}_n(x+a) = a + \mathcal{N}_n(x) = \{a + V : V \in \mathcal{N}_n(x)\}$$

Therefore,

$$\begin{aligned} E_a(L_n(f))(x) &= L_n(f)(x+a) = \max_{V \in \mathcal{N}_n(x+a)} \min_{y \in V} f(y) \\ &= \max_{V \in \mathcal{N}_n(x)} \min_{y \in a+V} f(y) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y+a) \\ &= \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} E_a(f)(y), \quad x \in \mathbb{Z}^d \end{aligned}$$

(ii) Let $f, c \in \mathcal{A}(\mathbb{Z}^d)$, where c is a constant function with a value of θ . Then for every $x \in \mathbb{Z}^d$ we have

$$\begin{aligned} L_n(f+c)(x) &= \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} (f+c)(y) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} (f(y) + \theta) \\ &= \left(\max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y) \right) + \theta = L_n(f)(x) + c(x) \end{aligned}$$

(iii) Let $f \in \mathcal{A}(\mathbb{Z}^d)$ and $\alpha \in \mathbb{R}$, $\alpha \geq 0$. For every $x \in \mathbb{Z}^d$ we have

$$L_n(\alpha f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{q \in V} (\alpha f)(q) = \alpha \left(\max_{V \in \mathcal{N}_n(x)} \min_{q \in V} f(q) \right) = \alpha L_n(f)(x).$$

(iv) The inequality

$$L_n \circ L_n \leq L_n$$

is an immediate consequence of Theorem 5. Then it is sufficient to prove the inverse inequality. Let $f \in \mathcal{A}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. We have

$$L_n(L_n(f))(x) = \max_{W \in \mathcal{N}_n(x)} \min_{y \in W} \max_{V \in \mathcal{N}_n(y)} \min_{z \in V} f(z). \quad (13)$$

But $y \in W \in \mathcal{N}_n(x)$ implies $W \in \mathcal{N}_n(y)$. Therefore for every $W \in \mathcal{N}_n(x)$ and $y \in W$ we have

$$\max_{V \in \mathcal{N}_n(y)} \min_{z \in V} f(z) \geq \min_{z \in W} f(z).$$

Using that the right hand side is independent of y we further obtain

$$\min_{y \in W} \max_{V \in \mathcal{N}_n(y)} \min_{z \in V} f(z) \geq \min_{z \in W} f(z), \quad W \in \mathcal{N}_n(x).$$

Then it follows from the representation (13) that

$$L_n(L_n(f))(x) \geq \max_{W \in \mathcal{N}_n(x)} \min_{z \in W} f(z) = L_n(f)(x).$$

(v) The co-idempotence of the operator L_n is equivalent to $L_n \circ (id - L_n) = 0$. The inequality $L_n \circ (id - L_n) \geq 0$ is an easy consequence of Theorem 5. Hence, for the co-idempotence of L_n it remains to show that $L_n \circ (id - L_n) \leq 0$. Assume the opposite. Namely, there exists a function $f \in \mathcal{A}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$ such that $(L_n \circ (id - L_n))(f)(x) > 0$. Using the definition of L_n this inequality implies that there exists $V \in \mathcal{N}_n(x)$ such that for every $y \in V$ we have $(id - L_n)(f)(y) > 0$, or equivalently

$$f(y) > L_n(f)(y), \quad y \in V. \quad (14)$$

Let $z \in V$ be such that $f(z) = \min_{t \in V} f(t)$. Then for every $y \in V$ we have

$$L_n(f)(y) = \max_{W \in \mathcal{N}_n(y)} \min_{t \in W} f(t) \geq \min_{t \in V} f(t) \geq f(z). \quad (15)$$

Taking $y = z$ in (14) and (15) we obtain a contradiction which completes the proof. ■

3 The operators L_n and U_n as smoothers

Similar to their counterparts for sequences the operators L_n and U_n defined in Section 2 smooth the input function by removing sharp peaks (the application of L_n) and deep pits (the application of U_n). The smoothing effect of these operations is made more precise by using the concepts of a local maximum set and a local minimum set given below.

Definition 7 Let $V \in \mathcal{C}$. A point x is called **adjacent** to V if $V \cup \{x\} \in \mathcal{C}$. The set of all points adjacent to V is denoted by $\text{adj}(V)$, that is,

$$\text{adj}(V) = \{x \in \mathbb{Z}^d : V \cup \{x\} \in \mathcal{C}\}.$$

An equivalent formulation of the property (8) of the connection \mathcal{C} is as follows:

$$V, W \in \mathcal{C}, W \subsetneq V \implies \text{adj}(W) \cap V \neq \emptyset. \quad (16)$$

Definition 8 A connected subset V of \mathbb{Z}^d is called a **local maximum set** of $f \in \mathcal{A}(\mathbb{Z}^d)$ if

$$\max_{y \in \text{adj}(V)} f(y) < \min_{x \in V} f(x).$$

Similarly V is a **local minimum set** if

$$\min_{y \in \text{adj}(V)} f(y) > \max_{x \in V} f(x).$$

The next four theorems deal with different aspects of the application of L_n and U_n to functions in $\mathcal{A}(\mathbb{Z}^d)$. Their cumulative effect will be discussed at the end of the section. All theorems contain statements a) and b). Due to the similarity we present only the proofs of a).

Theorem 9 Let $f \in \mathcal{A}(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. Then we have

- a) $L_n(f)(x) < f(x)$ if and only if there exists a local maximum set V such that $x \in V$ and $\text{card}(V) \leq n$;
- b) $U_n(f)(x) > f(x)$ if and only if there exists local minimum set V such that $x \in V$ and $\text{card}(V) \leq n$.

Proof. a) Implication to the left. Suppose that there exists a local maximum set $V \in \mathcal{N}_k(x)$, $k < n$. Consider an arbitrary $W \in \mathcal{N}_n(x)$ and let $S = W \cap V$. Then, since the size of W is larger than the size of S we have $W \setminus S \neq \emptyset$. Furthermore, by (16) we have $\text{adj}(S) \cap W \neq \emptyset$. Let $z \in \text{adj}(S) \cap W$. Since $\text{adj}(S) \cap W \subseteq W \setminus S = W \setminus V$, we have that $z \notin V$ but $z \in \text{adj}(V)$. Then using also that V is a local maximum set we obtain

$$\min_{y \in W} f(y) \leq f(z) < \min_{t \in V} f(t) \leq f(x).$$

Since the set $W \in \mathcal{N}_n(p)$ is arbitrary, this inequality implies that $L_n(f)(x) < f(x)$.

Implication to the right. Suppose $L_n(f)(x) < f(x)$. Let V be the largest (in terms of \subseteq) connected set containing x such that

$$f(y) \geq f(x), \quad y \in V. \quad (17)$$

The set V is obviously unique and can be constructed as $V = \gamma_x(Y)$, where γ_x is the morphological point connected opening generated by x , see [19] or [20], and $Y = \{y \in \mathbb{Z}^d : f(y) \geq f(x)\}$. We have $f(z) < f(x)$, $z \in \text{adj}(V)$, because otherwise (17) is satisfied on the larger connected set $\{z\} \cap V$. Therefore

$$\max_{z \in \text{adj}(V)} f(z) < f(x) = \min_{y \in V} f(y).$$

Hence V is a local maximum set.

Assume that $\text{card}(V) > n$. It follows from (10) that there exists $W \in \mathcal{N}_n(x)$ such that $W \subset V$. Then

$$L_n(f)(x) = \max_{V \in \mathcal{N}_n(x)} \min_{y \in V} f(y) \geq \min_{y \in W} f(y) \geq \min_{y \in V} f(y) = f(x).$$

This contradicts the assumption $L_n(f)(x) < f(x)$. Therefore, $\text{card}(V) \leq n$. ■

Theorem 10 *Let $f \in \mathcal{A}(\mathcal{Z}^d)$. Then*

- a) *the size of any local maximum set of the function $L_n(f)$ is larger than n ;*
- b) *the size of any local minimum set of the function $U_n(f)$ is larger than n .*

Proof. a) Assume the opposite, that is, there exists a local maximum set V of $L_n(f)$ such that $\text{card}(V) \leq n$. By Theorem 9 we have that

$$L_n(L_n(f))(x) < L_n(f)(x), \quad x \in V.$$

Since L_n is idempotent, see Theorem 6, this implies the impossible inequality $L_n(f)(x) < L_n(f)(x)$, which completes the proof. ■

Theorem 11 *Let $V \in \mathcal{C}$ and let $x \in \text{adj}(V)$.*

- a) *If $f(x) \leq \min_{y \in V} f(y)$ then $L_n(f)(x) \leq \min_{y \in V} L_n(f)(y)$;*
- b) *If $f(x) \geq \max_{y \in V} f(y)$ then $U_n(f)(x) \geq \max_{y \in V} U_n(f)(y)$.*

Proof. a) For any $W \in \mathcal{N}_n(x)$ the set $W \cup V$ is connected and of size larger than $n + 1$. Therefore, by (10), for every $y \in V$ there exists $S_y \in \mathcal{N}_n(y)$ such that $S_y \subset W \cup V$. Then, using also the given inequality, for every $y \in V$ and $W \in \mathcal{N}_n(x)$ we have

$$\min_{z \in W} f(z) = \min_{z \in W \cup V} f(z) \leq \min_{z \in S_y} f(z) \leq L_n(f)(y).$$

Hence

$$L_n(f)(x) = \max_{W \in \mathcal{N}_n(x)} \min_{z \in W} f(z) \leq \min_{y \in V} L_n(f)(y).$$

■

Theorem 12 *Let $f \in \mathcal{A}(\mathcal{Z}^d)$ and $V \in \mathcal{C}$.*

- a) *If V is a local minimum set of $L_n(f)$ then there exists a local minimum set W of f such that $W \subseteq V$.*
- b) *If V is a local maximum set of $U_n(f)$ then there exists a local maximum set W of f such that $W \subseteq V$.*

Proof. a) Let V be a local minimum set of $L_n(f)$. Then

$$\min_{y \in \text{adj}(V)} f(y) \geq \min_{y \in \text{adj}(V)} L_n(f)(y) > L_n(f)(x) \quad \forall x \in V.$$

Let $q \in \text{adj}(V)$ be such that $f(q) = \min_{y \in \text{adj}(V)} f(y)$ and let

$$Y = \{y \in V : f(y) < f(q)\}.$$

An easy application of Theorem 11 shows that $Y \neq \emptyset$. Let $t \in Y$ and let W be the largest (with respect to inclusion) connected subset of Y which contains t . As in the proof of Theorem 9, the set W can be obtained through $W = \gamma_t(Y)$. For every $z \in \text{adj}(W)$ we have $f(z) \geq f(q) > \max_{y \in W} f(y)$. Therefore W is a local minimum set of f . ■

Theorems 9–12 provide the following characterization of the effect of the operators L_n and U_n of a function $f \in \mathcal{A}(\mathbb{Z}^d)$:

- The application of L_n (U_n) removes local maximum (minimum) sets of size smaller or equal to n .
- The operator L_n (U_n) does not affect the local minimum (maximum) sets in the sense that such sets may be affected only as a result of the removal of local maximum (minimum) sets. However, no new local minimum sets are created where there were none. This does not exclude the possibility that the action of L_n (U_n) may enlarge existing local maximum (minimum) sets or join two or more local maximum (minimum) sets of f into one local maximum (minimum) set of $L_n(f)$ ($U_n(f)$).
- $L_n(f) = f$ ($U_n(f) = f$) if and only if f does not have local maximum (minimum) sets of size n or less;

Furthermore, as an immediate consequence of Theorem 10 and Theorem 12 we obtain the following corollary.

Corollary 13 *For every $f \in \mathcal{A}(\mathbb{Z}^d)$ the functions $(L_n \circ U_n)(f)$ and $(U_n \circ L_n)(f)$ have neither local maximum sets nor local minimum sets of size n or less. Furthermore,*

$$(L_n \circ U_n)(f) = (U_n \circ L_n)(f) = f$$

if and only if f does not have local maximum sets or local minimum sets of size less than or equal to n .

We should remark that in the one dimensional setting, the sequences without local maximum sets or local minimum sets of size less than or equal to n are exactly the so-called n -monotone sequences. Hence Corollary 13 generalizes the respective results in the LULU theory of sequences, [13, Theorem 3.3].

4 The LULU semi-group

In this section we consider the operators L_n , U_n and their compositions. The main result is that L_n , U_n , $L_n \circ U_n$ and $U_n \circ L_n$ form a semi-group with respect to composition with a composition table as given in Table 1. Furthermore, the semi-group is totally ordered as in (4) with respect to the point-wise defined partial order (3).

Theorem 14 *The operators $L_n \circ U_n$ and $U_n \circ L_n$ are idempotent, that is,*

$$L_n \circ U_n \circ L_n \circ U_n = L_n \circ U_n , \quad (18)$$

$$U_n \circ L_n \circ U_n \circ L_n = U_n \circ L_n . \quad (19)$$

Proof. Using the order properties in Theorem 5 and the idempotence of L_n and U_n , see Theorem 6, we have

$$\begin{aligned} L_n \circ U_n \circ L_n \circ U_n &\leq L_n \circ U_n \circ id \circ U_n = L_n \circ U_n \circ U_n = L_n \circ U_n \\ L_n \circ U_n \circ L_n \circ U_n &\geq L_n \circ id \circ L_n \circ U_n = L_n \circ L_n \circ U_n = L_n \circ U_n \end{aligned}$$

which implies (18). The equality (19) is proved similarly. ■

Theorem 15 *For any $n \in \mathbb{N}$ we have*

$$L_n \circ U_n \circ L_n = U_n \circ L_n. \quad (20)$$

Proof. It follows from Theorem 5 that

$$L_n \circ U_n \circ L_n \leq id \circ U_n \circ L_n = U_n \circ L_n. \quad (21)$$

Assume that (20) is violated. In view of (21), this means that there exists $f \in \mathcal{A}(\mathbb{Z}^d)$ and $z \in \mathbb{Z}^d$ such that

$$L_n(U_n(L_n(f)))(z) < U_n(L_n(f))(z).$$

It follows from Theorem 9 that there exists $k \leq n$ and $V \in \mathcal{N}_k(z)$ such that V is a local maximum set for $U_n(L_n(f))(z)$. Then, by Theorem 12, there exists $W \subseteq V$ such that W is a local maximum set of the function $L_n(f)$. We have $\text{card}(W) \leq k \leq n$. However, $L_n(f)$ does not have any local maximum sets of size less than or equal to n , see Theorem 10. This contradiction completes the proof. ■

As in the case of sequences, the key result for the set

$$\{L_n, U_n, L_n \circ U_n, U_n \circ L_n\} \quad (22)$$

to be closed under composition is the equality in Theorem 15. Now one can easily derive the rest of the formulas for the compositions of the operators in this set. The composition table is indeed as given in Table 1. Furthermore, Theorem 15 implies the total order on the set (22) as in (4). Indeed, we have

$$L_n = id \circ L_n \leq U_n \circ L_n = L_n \circ U_n \circ L_n \leq L_n \circ U_n \circ id = L_n \circ U_n \leq id \circ U_n = U_n$$

Therefore, the operators L_n and U_n for functions on \mathbb{Z}^d generate via composition a semi-group with exactly the same algebraic and order structure as the semi-group generated by the operators L_n and U_n for sequences.

5 Discrete pulse transform of images

In this section we apply the LULU operators defined and investigated in the preceding sections to derive a discrete pulse decomposition of images. A grayscale image is given through a function f on a rectangular domain $\Omega \subset \mathbb{Z}^2$, the value of f being the luminosity at the respective pixel. For the theoretical study it is more convenient to assume that the functions are defined on the whole space \mathbb{Z}^2 . To this end one can, for example, define f on the set $\mathbb{Z}^2 \setminus \Omega$ as a constant, e.g. 0. Hence we consider the set $\mathcal{A}(\mathbb{Z}^2)$.

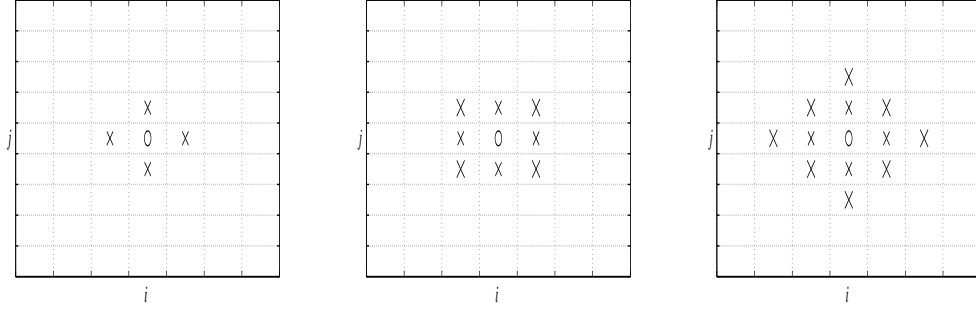


Figure 1: Neighbors of (i, j)

Appropriate connections for images are defined through a relation r on \mathbb{Z}^2 reflecting what we consider neighbors of a pixel in the given context. Figure 1 gives some examples of the the neighbors of the pixel (i, j) .

We call a set $C \subseteq \mathbb{R}^2$ *connected* if for any two pixels $p, q \in C$ there exists a set of pixels $\{p_1, p_2, \dots, p_k\} \subseteq C$ such that each pixel is neighbor to the next one, p is neighbor to p_1 and p_k is neighbor to q . We assume that the neighbor relation r on \mathbb{Z}^2 is such that

- r is reflexive, symmetric and shift invariant (23)

- $((i, j), (i \pm 1, j)) \in r$ and $((i, j), (i, j \pm 1)) \in r$, for all $i, j \in \mathbb{Z}$. (24)

The conditions (23)–(24) ensure that the set of connected set \mathcal{C} defined through this relation is a connection in terms of Definition 3 and satisfies the conditions (6)–(8). Hence we can apply the operators L_n and U_n discussed in the preceding sections to functions on \mathbb{Z}^2 . Similar to the case of sequences we obtain a decomposition of a function $f \in \mathcal{A}(\mathbb{Z}^2)$ by applying iteratively the operators L_n, U_n with n increasing from 1 to ∞ . This can be done in different ways depending on sequencing of the L_n and U_n . Since this section is intended as a demonstration rather than presenting a comprehensive discrete pulse transform theory, we will take one particular case when U_n follows L_n . Define the operators F_n , $n \in \mathbb{N}$, by $F_1 = U_1 \circ L_1$ and $F_n = U_n \circ L_n \circ F_{n-1}$. Then for any $f \in \mathcal{A}(\mathbb{Z}^2)$ and $m \geq 1$ we have

$$\begin{aligned} f &= (id - U_1 \circ L_1)(f) + ((id - U_2 \circ L_2) \circ F_1)(f) + ((id - U_3 \circ L_3) \circ F_2)(f) \\ &\quad + \dots + ((id - U_m \circ L_m) \circ F_{m-1})(f) + F_m(f) \end{aligned} \quad (25)$$

Definition 16 A function $\phi \in \mathcal{A}(\mathbb{Z}^2)$ is called a pulse if there exist a connected set V and a real number α such that

$$\phi(x) = \begin{cases} \alpha & \text{if } x \in V \\ 0 & \text{if } x \in \mathbb{Z}^2 \setminus V. \end{cases}$$

The set V is called support of the pulse ϕ and is denoted by $\text{supp}(\phi)$.

Figure 2 gives an example of a pulse. It should be remarked that the support of a pulse may generally have any shape, the only restriction being that it is connected.

The usefulness of the representation (25) of a function $f \in \mathcal{A}(\mathbb{Z}^2)$ is in the fact that all terms are sums of pulses as stated in the next theorem.

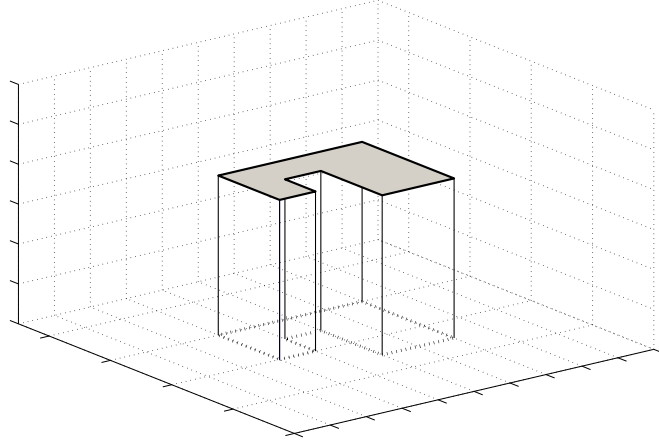


Figure 2: Pulse

Theorem 17 *Let $f \in \mathcal{A}(\mathbb{Z}^2)$.*

- a) *For every $n \in \mathbb{N}$ the function $((id - U_n \circ L_n) \circ F_{n-1})(f)$ is a sum of discrete pulses with disjoint support, that is, there exist $\gamma(n) \in \mathbb{N}$ and discrete pulses ϕ_{ns} , $s = 1, \dots, \gamma(n)$, such that*

$$((id - U_n \circ L_n) \circ F_{n-1})(f) = \sum_{s=1}^{\gamma(n)} \phi_{ns} \quad (26)$$

and

$$\text{supp}(\phi_{ns_1}) \cap \text{supp}(\phi_{ns_2}) = \emptyset \text{ for } s_1 \neq s_2. \quad (27)$$

- b) *Let $n_1, n_2, s_1, s_2 \in \mathbb{N}$ be such that $n_1 < n_2$, $1 \leq s_1 \leq \gamma(n_1)$ and $1 \leq s_2 \leq \gamma(n_2)$. Then*

$$\text{supp}(\phi_{n_1 s_1}) \cap \text{supp}(\phi_{n_2 s_2}) \neq \emptyset \implies \text{supp}(\phi_{n_1 s_1}) \subset \text{supp}(\phi_{n_2 s_2}) \quad (28)$$

Proof. a) Denote $g = F_{n-1}(f)$. We have

$$((id - U_n \circ L_n) \circ F_{n-1})(f) = (id - L_n)(g) + (id - U_n)(L_n(g)), \quad (29)$$

where the first term in the sum on the right hand side is nonnegative while the second one is nonpositive. Let $x \in \mathbb{Z}^2$ be such that $((id - L_n)(g)) > 0$. It follows from Theorem 9 that there exists a local maximum set V of g such that $x \in V$ and $\text{card}(V) \leq n$. Since $g = (U_{n-1} \circ L_{n-1})(F_{n-2}(f))$ does not have local maximum set of size smaller than n , see Corollary 13, this implies that $\text{card}(V) = n$ and that g is a constant on V . Furthermore, $((id - L_n)(g))(y) = 0$ for $y \in \text{adj}(V)$. Indeed, if $((id - L_n)(g))(y) > 0$ for some $y \in \text{adj}(V)$, then y belongs to a local maximum set W of g and $\text{card}(W) \leq n$. However, any maximum set containing y must contain V as well which implies $\text{card}(W) \geq n + 1$, a contradiction. In this way we obtain that the support of $(id - L_n)(g)$ is a union of disjoint connected sets of size n , that is,

$$\text{supp}((id - L_n)(g)) = V_1 \cup V_2 \cup \dots \cup V_{\gamma(n)},$$

where $V_s \in \mathcal{C}$, $\text{card}(V_s) = n$, $s = 1, \dots, \gamma_1(n)$ and $V_{s_1} \cap V_{s_2} = \emptyset$ for $s_1 \neq s_2$. Furthermore, $(id - L_n)(g)$ is a constant on each set V_s . If $(id - L_n)(g)(x) = \alpha_s$ for $x \in V_s$, then

$$(id - L_n)(g) = \sum_{s=1}^{\gamma_1(n)} \phi_{ns}, \quad (30)$$

where

$$\phi_{ns}(x) = \begin{cases} \alpha_s & \text{if } x \in V_s \\ 0 & \text{if } x \in \mathbb{Z}^2 \setminus V_s \end{cases}$$

Applying the same approach to the second term in (29) we obtain

$$\text{supp}((id - L_n)(g)) = W_1 \cup W_2 \cup \dots \cup W_{\gamma_2(n)},$$

where $W_s \in \mathcal{C}$, $\text{card}(W_s) = n$, $s = 1, \dots, \gamma_2(n)$, $W_{s_1} \cap W_{s_2} = \emptyset$ for $s_1 \neq s_2$ and

$$(id - U_n)(L_n(g)) = \sum_{s=\gamma_1(n)+1}^{\gamma(n)} \phi_{ns}, \quad (31)$$

where $\gamma(n) = \gamma_1(n) + \gamma_2(n)$ and $\text{supp}(\phi_{ns}) = W_{s-\gamma_1(n)}$, $s = \gamma_1(n) + 1, \dots, \gamma(n)$. Note that ϕ_{ns} , $s = 1, \dots, \gamma_1(n)$, are upward (positive) pulses while ϕ_{ns} , $s = \gamma_1(n) + 1, \dots, \gamma(n)$ are downward (negative) pulses. We obtain (26) by substituting (30) and (31) in (29). It only remains to show that $V_{s_1} \cap W_{s_2} = \emptyset$ for all $s_1 = 1, \dots, \gamma_1(n)$, $s_2 = 1, \dots, \gamma_2(n)$. Indeed, assume that $V_{s_1} \cap W_{s_2} \neq \emptyset$. From the definition of the operator L_n , there exists $y \in \text{adj}(V_{s_1})$ such that $L_n(g)(x) = g(y)$ for $x \in V_{s_1} \cap \{y\}$. Therefore, $V_{s_1} \cap \{y\} \subseteq W_{s_2}$, which implies that $\text{card}(W_{s_2}) \geq n + 1$. Since the size of each one of the sets $W_1, \dots, W_{\gamma_2(n)}$ is n , none of them intersects V_{s_1} .

b) Let $\text{supp}(\phi_{n_1 s_1}) \cap \text{supp}(\phi_{n_2 s_2}) \neq \emptyset$. It follows from the construction of (26) derived in a) that the functions $F_n(f)$ and $L_{n+1}(F_n(f))$, $n \geq n_1$, are constants on the set $\text{supp}(\phi_{n_1 s_1})$. Furthermore, the set $\text{supp}(\phi_{n_2 s_2})$ is a local maximum set of $F_{n_2-1}(f)$ or a local minimum set of $L_{n_2}(F_{n_2-1}(f))$. From the definition of local maximum set and local minimum set it follows that $\text{supp}(\phi_{n_1 s_1}) \subset \text{supp}(\phi_{n_2 s_2})$. ■

Using Theorem 17, the equality (25) can be written in the form

$$f = \sum_{k=1}^m \sum_{s=1}^{\gamma(k)} \phi_{ks} + F_m(f). \quad (32)$$

If the function f has finite support, e.g. as in the case of images, then $F_m(f)$ is a constant for a sufficiently large m . Then we have

$$f = \sum_{k=1}^m \sum_{s=1}^{\gamma(k)} \phi_{ks} + c, \quad (33)$$

where $c = F_m(f)(x)$, $x \in \mathbb{Z}^2$. The equality (33) is a discrete pulse decomposition of f , where the pulses have the properties (27)–(28). It is generally accepted that an image is perceived through the contrast, that is, the difference in the luminosity of neighbor pixels. The discrete pulse transform (33) extracts all such differences as single pulses. Hence, (33) can be a useful tool

in the analysis of images. Since the information in an image is in the contrast, the total variation of the luminosity function is an important measure of the quantity of this information. Image recovery and noise removal via total variation minimization are discussed in [3] and [16]. It should be noted that there are several definition of total variation of functions of multi-dimensional argument (Arzel variation, Vitali variation, Pierpont variation, Hardy variation, etc.). In the applications cited above the total variation is the L^1 norm of a vector norm of the gradient of the function. Here we consider a discrete analogue of this concept.

Definition 18 Let $f \in \mathcal{A}(\mathbb{Z}^2)$. If

$$\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (|f(i+1, j) - f(i, j)| + |f(i, j+1) - f(i, j)|) < \infty \quad (34)$$

then f is said to be of bounded variation. The sum on the left side on the inequality (34) is called total variation of f and is denoted by $TV(f)$.

As mentioned in the introduction, the LULU operators for sequences are total variation preserving. We show here that their two-dimensional counterparts considered in this section have the same property with respect to the total variation as given in Definition 18.

Let us denote by $BV(\mathbb{Z}^2)$ the set of all functions of bounded variation in $\mathcal{A}(\mathbb{Z}^2)$. Clearly, all functions of finite support are in $BV(\mathbb{Z}^2)$. In particular, the luminosity functions of images are in $BV(\mathbb{Z}^2)$. The total variation given in Definition 18 is a semi-norm on $BV(\mathbb{Z}^2)$. In particular, this implies that

$$TV(f + g) \leq TV(f) + TV(g). \quad (35)$$

The total preservation property is defined for operators on $BV(\mathbb{Z}^2)$ as in Definition 2, where \mathbb{Z} is replaced by \mathbb{Z}^2 .

Theorem 19 The operators L_n , U_n , $n=1,2,\dots$, and their compositions are all total variation preserving.

Proof. Let $f \in BV(\mathbb{Z}^2)$ and $(i, j) \in \mathbb{Z}^2$. We will show that

$$\begin{aligned} |f(i, j) - f(i+1, j)| &= |L_n(f)(i, j) - L_n(f)(i+1, j)| \\ &\quad + |(id - L_n)(f)(i, j) - (id - L_n)(f)(i+1, j)| \end{aligned} \quad (36)$$

Case 1. $L_n(f)(i, j) < f(i, j)$. It follows from Theorem 9 that there exists a local maximum set V such that $(i, j) \in V$ and $\text{card}(V) \leq n$. Without loss of generality we may assume that V is the largest set with the said properties. Then $L_n(f)(x) = f(z)$, $x \in V$, where $z \in \text{adj}(V)$ is such that $f(z) = \max_{y \in \text{adj}(V)} (f)(y)$. Since $(i+1, j)$ is a neighbor to (i, j) , see (24), we have either $(i+1, j) \in V$ or $(i+1, j) \in \text{adj}(V)$.

Case 1.1 $(i+1, j) \in V$. Then $L_n(f)(i, j) - L_n(f)(i+1, j) = f(z) - f(z) = 0$ and (36) trivially holds.

Case 1.2 $(i+1, j) \in \text{adj}(V)$. Then $(i+1, j)$ cannot be element of a local maximum set of size smaller or equal to n . Therefore, $L_n(f)(i+1, j) = f(i+1, j) \leq f(z) = L_n(f)(i, j)$, which implies (36).

Case 2. $L_n(f)(i, j) = f(i, j)$. If $L_n(f)(i + 1, j) = f(i + 1, j)$ the equality (36) trivially holds. If $L_n(f)(i + 1, j) < f(i + 1, j)$, then we obtain (36) by repeating the argument in Case 1.2 where the points (i, j) and $(i + 1, j)$ change places.

Similarly to (36) we prove that

$$\begin{aligned} |f(i, j) - f(i, j + 1)| &= |L_n(f)(i, j) - L_n(f)(i, j + 1)| \\ &\quad + |(id - L_n)(f)(i, j) - (id - L_n)(f)(i, j + 1)| \end{aligned}$$

Then by Definition 18 we have

$$TV(f) = TV(L_n(f)) + TV((id - L_n)(f)).$$

The total variation preserving property of U_n is proved in a similar way.

In order to complete the proof we show that the composition $A \circ B$ of any two total variation preserving operators A and B on $BV(\mathbb{Z}^2)$ is also total variation preserving. Using the total variation preserving property of A and B and (35) we have

$$\begin{aligned} TV(f) &= TV(B(f)) + TV((id - B)(f)) \\ &= TV(A(B(f))) + TV((id - A)(B(f))) + TV((id - B)(f)) \\ &\geq TV((A \circ B)(f)) + TV(((id - A) \circ B + id - B)(f)) \\ &= TV((A \circ B)(f)) + TV((id - A \circ B)(f)). \end{aligned}$$

From (35) we also obtain $TV(f) \leq TV((A \circ B)(f)) + TV((id - A \circ B)(f))$. Therefore $TV(f) = TV((A \circ B)(f)) + TV((id - A \circ B)(f))$. ■

Let function $f \in \mathcal{A}(\mathbb{Z}^2)$ have finite support, e.g. as in the case of images. Then $f \in BV(\mathbb{Z}^2)$. Using Theorem 19 the discrete pulse decomposition (33) is total variation preserving in the sense that

$$TV(f) = \sum_{k=1}^m \sum_{s=1}^{\gamma(k)} TV(\phi_{ks}). \quad (37)$$

We should remark that representing a function as a sum of pulses can be done in many different ways. However, in general, such decompositions increase the total variation, that is, we might have strict inequality in (37) instead of equality. The equality in (37) means that no additional total variation, or noise, is created via the decomposition.

6 Partial reconstructions and noise removal

Possibly the simplest application of the discrete pulse decomposition (33) is via partial reconstructions of images. This can be used for example in removing noise or extracting features of interest. Random noise has very distinctive discrete pulse decomposition characterized by fast decrease of the number of pulses with the increase of the pulse size. The number of pulses in decomposition (33) versus their size for a 300×400 image of random noise (the luminosity at each pixel is an independent uniformly distributed random variable) is plotted on Figure 3. It is apparent that random noise seldom generates pulses of large size. In fact, 90% of the pulses represented on Figure

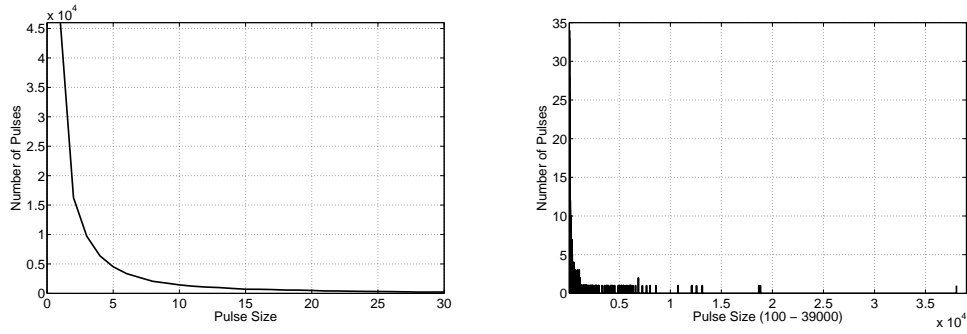


Figure 3: Pulse distribution of random noise



Figure 4: A sea image

3 are of size less than or equal to 20 and only about 2% have size greater than 100. Hence by removing the pulse of small support we remove large portion of any impulsive noise. Figure 5 gives in the same format the pulse distribution of the image on Figure 4. A large portion of the pulses has small support but, unlike Figure 3, we have also significant number of pulses with relatively larger support. Partial reconstruction of the image by using pulses of selected sizes is given on Figure 6. We can consider (a) as removing of impulsive noise, (b) as extraction of small features and (c) as extraction of large features.

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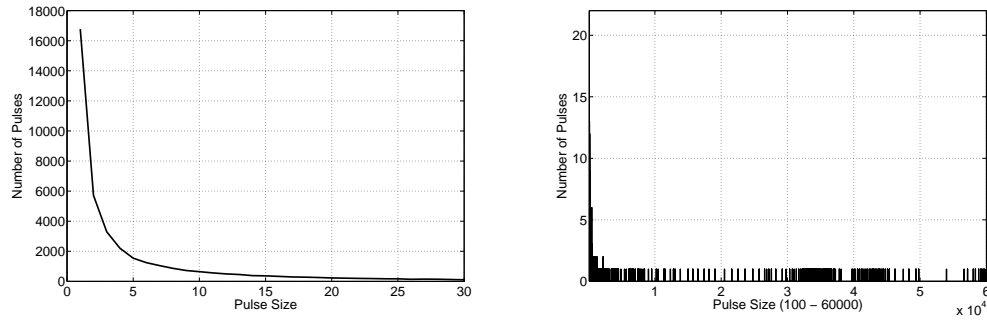


Figure 5: Pulse distribution of the sea image

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(a)



(b)



(c)



Figure 6: Partial reconstructions of the sea image: (a) Pulses of size larger than 20; (b) Pulses of size from 21 to 400; (c) Pulses of size from 30000 to 50000.